

Bosonic and fermionic behavior in gravitational configurations

Leonardo Patiño ^{*} and Hernando Quevedo [†]

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México

A. P. 70-543, México 04510 D.F., México

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We extend Dirac's approach about the quantization of the electric charge to the case of gravitational configurations. The spacetime curvature is used to define a phase-like object which allows us to extract information about the behavior of the corresponding spacetime. We show that all spacetimes that satisfy certain simple symmetry condition and for which the Petrov type is the same within a specific region, quantization conditions can be derived that impose constraints on the possible values of the parameters entering the respective metrics. As a general result we obtain that for the gravitational configurations described by those metrics, the behavior under rotations can be only of bosonic or fermionic nature.

I. INTRODUCTION

It is widely believed that half-integral spin objects (fermions) appear only in quantum theories and that they have no classical counter-partners. Objects with integral spin (bosons), instead, are characteristic in both classical and quantum configurations. One of the most important properties of these two kind of objects is the way how they behave under rotations: Bosons are invariant under a 2π rotation of space coordinates, while fermions are invariant only under a 4π rotation.

An argument in favor of the existence of a half-integral fermionic behavior in classical theories was given by Finkelstein and Misner [1]. Later on, this conjecture was successfully "tested" by Friedman and Sorkin [2] who analyzed the behavior under diffeomorphisms of a Schrödinger state vector in the configuration space defined by asymptotically flat 3-dimensional classical metrics. Although this approach is based upon very natural assumptions about the kinematic behavior of the gravitational state vector, a complete theory of quantum gravity will be necessary in order to confirm this result. More recently, Hadley [3] has shown that even at the level of classical general relativity a fermionic behavior naturally appears, when geons are treated as models for elementary particles. But the first indication that half-integral spins can arise as a composite state of integral spin constituents was given originally by Dirac in his seminal paper of 1931 [4]. Indeed, Dirac analyzed the quantum phase acquired by a charged particle when dragged around a closed loop in the presence of a magnetic monopole. In this case, the charge-monopole system can have half-integral spin, even if the electric charge and the magnetic monopole by themselves have integral spin. One further important result of this analysis is that it predicts the quantization of the electric charge, if the magnetic monopole exists.

In this work, we will follow Dirac's original idea which we now briefly review. The quantum phase acquired by a charged particle while moving within a magnetic field B is given by

$$\Phi = e^{iq \int_{\gamma} A}, \quad (1)$$

where A is the vector potential for B . *i.e.* ($dA = B$), γ is the path followed by the particle and q is its charge. If γ is a closed path, we can use Stoke's theorem to get

$$\Phi = e^{iq \int_S B}, \quad (2)$$

with S any surface having γ as its boundary. The magnetic field of a magnetic monopole in spherical coordinates is $B = g \sin \theta \, d\theta \wedge d\phi$, where g is the magnetic charge. The interesting feature of this field is that it is a closed two-form which is not exact. Therefore, if we have a charged particle immersed in this field, and we drag it along a circle around

^{*}E-mail address: leonardo@ft.ifisicacu.unam.mx

[†]E-mail address: quevedo@nuclecu.unam.mx

the origin of the coordinate system, we can use Eq.(2) to calculate the phase, but we cannot use Eq.(1), because there is no such an A . In order to calculate the phase (2) we can use different surfaces, in particular we can use the two different hemispheres which have γ as their boundary, and γ is usually taken as a circle on the equatorial plane¹. The sign of the integral over one hemisphere must be the opposite of the sign of the integral over the other hemisphere in order to be consistent with the orientation of γ . Then we get $\Phi = \exp(\frac{-iqg}{2})$ for one hemisphere and $\Phi = \exp(\frac{iqg}{2})$ for the other one. If we demand that these two results be equal, we get that $\frac{qg}{2} = n\pi$, for any integer n . Dirac concluded from here that the existence of a magnetic monopole will imply the quantization of the electric charge $q = n\frac{2\pi}{g}$.

In this work, we will apply Dirac's approach to the case of gravity. In Section 2, we introduce a phase-like object which explicitly contains a surface integral of the spacetime curvature. A brief discussion about some details of this definition is included. Section 3 is devoted to the investigation of the symmetries of the eigenvalues of the curvature tensor. We also introduce the symmetry of spacetime that is necessary in order to perform a quite general calculation of the phase-like object which is the subject of Section 4. In this Section, we also impose Dirac's quantization condition on our phase-like object and prove that it leads to the conclusion that classical gravitational configurations may behave either as bosons or as fermions under rotations and that, in some special cases, the parameters entering the spacetime metric have to satisfy certain quantum conditions. Finally, in Section 6 we present the conclusions of our results.

II. GENERALIZATION TO GRAVITATION

First, let us notice that the integral in Eq.(2) is in fact an integral of the electromagnetic tensor over a specific surface. In the special case of a magnetic monopole, the only non vanishing entrance of the electromagnetic tensor is the one corresponding to $d\theta \wedge d\phi$ in spherical coordinates. On the other hand, from the geometrical point of view of field theory the electromagnetic tensor is just the curvature associated to the electromagnetic connection. Consider a Riemannian manifold $(M, g_{\mu\nu})$, where $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) is the underlying metric. Then we can introduce a phase-like object²

$$\Phi = e^{\int R}, \quad (3)$$

where R is the curvature associated with the metric, namely the Riemann tensor, which is an endomorphism valued two-form (see, for instance, [5]). The integral in Eq.(3) has to be performed over the two-form part of the Riemann tensor, and the result is an endomorphism. So, the phase-like object (3), being the exponential of an endomorphism, is itself an endomorphism.

The Riemann tensor depends on the point of the manifold where it is evaluated, since it is a two-form which components are endomorphisms. These components actually live in the tangent and cotangent spaces to the manifold at the specific point of evaluation. If we perform the integral in Eq.(3) just as it appears, we will be adding objects which live in different spaces and this summation will not be justified. What we need is to perform a parallel translation of the endomorphism³ from the point of evaluation to a specific point where we will say that the integral is based. So, instead of (3) we should use

$$\Phi = e^{\int H^{-1}RH}, \quad (4)$$

where H is the holonomy resulting from the parallel transport, and it will depend explicitly on the point of evaluation. Of course, there are infinite many different paths along which the parallel transport can be performed. Therefore, the specific form of H will have to be fixed by using a different approach [6]. For the moment, the important point is that there are some conclusions that can be stated independently of the explicit form of the holonomy H . The only condition we will require below is that it preserves some fundamental symmetries of the curvature. Since H is directly related to the metric and its symmetries, it seems natural to expect this behavior. Otherwise, it can be imposed by construction.

¹Actually the circle must be vertical, otherwise the points $\theta = 0$ and $\theta = \pi$, which are in the boundary of the domain of integration, get mapped into the interior of the surface over the manifold.

²An exact definition would need, in general, a path-ordered exponential and a constant factor in front of the integral depending on the choice of units.

³This parallel translation has to be performed only over the endomorphism part of the Riemann tensor; the two-form will be handled using the pull-back in the conventional way.

The key point of the choice of surfaces in Dirac's argument is that the two hemispheres are not homotopic. So the physical relevant cases expected in the analysis of gravitational configurations will be those topological spaces which are solutions to the Einstein equations and accept the existence of not homotopic surfaces with common boundary, for instance, manifolds with localized curvature singularities.

III. SYMMETRIES

In order to analyze the phase-like object defined in the previous section for gravitational fields in a general fashion, we will consider the curvature tensor in its $SO(3, C)$ -representation [7]. Moreover, we will limit ourselves to *vacuum* solutions of Einstein's field equations⁴. The curvature tensor is represented by a 3×3 complex matrix \mathbf{Q} , the entrances of which are complex combinations of the components of the Riemann tensor written in a local orthonormal tetrad $\{e^a\}$, with local metric $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$. According to the Petrov classification, there exists a special orthonormal tetrad in which the matrix \mathbf{Q} takes a very simple form, the *normal* form. We will see that this normal form allows us to extract certain symmetries of the curvature which are crucial for the calculation of the phase-like object (4).

Consider, for instance, the normal form of a type *I* curvature tensor [7]

$$\mathbf{Q} = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}, \quad (5)$$

where τ_n ($n = 1, 2, 3$) are the eigenvalues of the curvature, which are subject to the constraint $\tau_1 + \tau_2 + \tau_3 = 0$, imposed by Einstein's vacuum field equations. For concrete calculations of the integral in Eq.(4), we need to know the explicit components of the curvature tensor. These components can be recovered from the normal form (5). For example, the components associated with the two-form $e^1 \wedge e^2$ can be expressed as the 4×4 real matrix⁵

$$R^a_{b12} = \begin{pmatrix} 0 & 0 & 0 & -Im(\tau_3) \\ 0 & 0 & -Re(\tau_3) & 0 \\ 0 & Re(\tau_3) & 0 & 0 \\ -Im(\tau_3) & 0 & 0 & 0 \end{pmatrix}, \quad (7)$$

where Re and Im stand for the real and imaginary part of the argument, respectively. Now, since the Riemann tensor is an endomorphism valued two-form, when we talk about its components, we refer to the endomorphisms associated with each entrance of the two-form. These components map a four dimensional space into itself, so they must have four eigenvalues λ_i . In the explicit example of Eq.(7), these eigenvalues are given by $\lambda_1 = -Im(\tau_3)$, $\lambda_2 = Im(\tau_3)$, $\lambda_3 = iRe(\tau_3)$, and $\lambda_4 = -iRe(\tau_3)$, and satisfy the relationships

$$\lambda_1 = -\lambda_2 \quad \text{and} \quad \lambda_3 = -\lambda_4. \quad (8)$$

Now, the important fact is that Eq.(8) holds for all the components of the curvature tensor. The explicit form of the matrix (7) may be different, but in all the cases the eigenvalues satisfy the symmetry property (8). Furthermore, it can be shown that this property is also valid for all the algebraically special curvature tensors, *i.e.*, for the Petrov types *D*, *II*, *N*, *III*, and *O*. That is to say, the relationships (8) constitute a universal property of vacuum gravitational fields. The importance of this result for the calculation of the phase (4) will be demonstrated in the next section.

The invariant character of the symmetry (8) follows from the fact that the Weyl principal tetrad $\{e^a\}$, in which the curvature tensor takes its normal form, is uniquely determined (up to trivial reflections) for the non degenerate Petrov

⁴For non vacuum gravitational fields, our approach can be applied with no changes to the Weyl tensor.

⁵We use the following convention: $\mathbf{Q} = \mathbf{A} + i\mathbf{B}$, where \mathbf{A} and \mathbf{B} are real 3×3 matrices which determine the 6×6 curvature matrix

$$\mathbf{R} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & -\mathbf{A} \end{pmatrix}. \quad (6)$$

All the independent components of the curvature tensor are contained in the matrix \mathbf{R} . Our convention for the transition between the matrix entrances of \mathbf{R} and the pair of tetrad indices is [8]: $1 \rightarrow 01$, $2 \rightarrow 02$, $3 \rightarrow 03$, $4 \rightarrow 23$, $5 \rightarrow 31$, $6 \rightarrow 12$.

types (*I*, *II*, and *III*). In the case of the degenerate types (*D*, *N*, and *O*), the Weyl principal tetrad is fixed only up to special Lorentz transformations [7] which, however, do not affect the invariant character of the curvature eigenvalues τ_n and, consequently, preserve the symmetry property (8). Moreover, it is diffeomorphism invariant because changes of coordinates do not affect the local tetrads. In the practice, a local change of coordinates implies a summation over a specific linear combination of the components of the curvature tensor. From the form of the Riemann tensor, as extracted from its normal form, it is not difficult to show that any linear combination of its components has eigenvalues satisfying Eq.(8).

The explicit calculation of (4) includes the holonomy H . Therefore, we have to guarantee that it does not affect the symmetry property (8). To do this, we first have to demand that the type in the Petrov classification of the curvature tensor does not change on any point of the path defined by H . Hence, we will limit ourselves to those regions of spacetime in which no changes of the Petrov type occur. This seems to be sufficient for satisfying Eq.(8) on all the points of the path, but since in this work we are not giving an exact definition of H , we prefer for the moment to impose the further condition that the holonomy H behaves in such a way that the eigenvalues of the endomorphism resulting from the integration satisfy Eq.(8).

Let us now describe the surfaces of integration. We can construct a closed surface symmetric with respect to an axis, say \mathbf{z} , and intersect it with any plane which contains this axis. In this way, we obtain two closed surfaces with the same boundary. If the spacetime is described by a Riemannian metric with a localized curvature singularity which is placed between the two surfaces, then the two surfaces are non homotopic. We now introduce spherical coordinates. Let us suppose that the spacetime metric is invariant with respect to the coordinate change⁶ $\phi \rightarrow \phi + \pi$. Now, by virtue of this symmetry, the integrals of the Riemann tensor over the two surfaces can differ only by a sign⁷. We will denote these two integrals by I_1 and I_2 , and the corresponding phases by $\Phi_1 = \exp(\pm I_1)$ and $\Phi_2 = \exp(\mp I_2)$, where the sign has to be chosen in accordance with the orientation of the boundary.

IV. CONDITION OF “QUANTIZATION”

Following Dirac’s argument we demand that the phase-like object (4) has the same value on both surfaces, *i. e.*, $\Phi_1 = \Phi_2$, a requirement that we call condition of “quantization”. If the integrals I_1 and I_2 have opposite signs, then the phase-like objects Φ_1 and Φ_2 obtained from them are equal, since to be consistent with the orientation of the boundary one of the signs must be reversed before exponentiating. In this situation the requirement $\Phi_1 = \Phi_2$ is trivially satisfied, and no additional condition can be obtained from the analysis.

On the other hand, if $I_1 = I_2$, then due to the change of sign that has to be done in one of these two integrals, say $-I_2$, before exponentiating, Φ_1 and Φ_2 will be different and, consequently, a further analysis is required. In this case, the eigenvalues of I_1 differ from the eigenvalues of $-I_2$ only by a minus sign. This together with the symmetry property (8) implies that both sets of eigenvalues coincide and the only difference appears in the interchange of the eigenvectors associated with each of these eigenvalues. Now we have to exponentiate the integrals I which are in general 4×4 matrices following from the integration of each of the entrances of the Riemann tensor as given, for example, in Eq.(7).

A simple way to deal with the exponential of a matrix, say M , is first to diagonalize it and then to calculate the exponential as

$$\exp(M) = T \exp(D) T^{-1}, \quad (9)$$

where D is the diagonal matrix which has the eigenvalues of M as components, and the matrix T represents the change of basis and is constructed explicitly by placing the eigenvectors of M as columns in the same order as the associated eigenvalues appear in D . Following this procedure for Φ_1 and Φ_2 , we see that since I_1 and $-I_2$ have the same eigenvalues λ_i satisfying the symmetry property (8), we can define a matrix

$$A \equiv \exp \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & -\lambda_3 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 & 0 & 0 \\ 0 & e^{-\lambda_1} & 0 & 0 \\ 0 & 0 & e^{\lambda_3} & 0 \\ 0 & 0 & 0 & e^{-\lambda_3} \end{pmatrix}, \quad (10)$$

⁶Clearly, this condition is satisfied by a large class of gravitational configurations, for instance, all the axially symmetric solutions.

⁷The difference in the sign comes from the fact that when the metric is symmetric with respect to some transformation, the Riemann tensor can be either symmetric or antisymmetric with respect to it.

and then express

$$\Phi_1 = T_1 A T_1^{-1}, \quad \Phi_2 = T_2 A T_2^{-1}. \quad (11)$$

From Eq.(11) it follows that if we impose the condition of quantization

$$\Phi_1 = \Phi_2 \Leftrightarrow [A, T_2^{-1} T_1] = 0, \quad (12)$$

where square brackets denote the commutator of the corresponding matrices. Thus, the verification of the vanishing of this commutator is a very easy way to extract information from the quantization condition $\Phi_1 = \Phi_2$.

As we have mentioned above, the eigenvectors of I_1 and I_2 are the same; the only differences is that the associated eigenvalues are interchanged. Therefore, the matrices T_1 and T_2 have the same columns in different order. Then, the general result is

$$T_2^{-1} T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (13)$$

From here and Eq.(10) we see that the commutator in Eq.(12) will vanish only if

$$e^{\lambda_1} = e^{-\lambda_1} \quad \text{and} \quad e^{\lambda_3} = e^{-\lambda_3}, \quad (14)$$

an equation that leads to the conditions

$$\lambda_1 = i n_1 \pi \quad \text{and} \quad \lambda_3 = i n_2 \pi, \quad (15)$$

where n_1 and n_2 are arbitrary integers. Remember that the eigenvalues λ_i depend on the parameters of the metric. Then, Eqs.(15) can be interpreted as quantum conditions on those parameters⁸, similar to that obtained by Dirac. Some concern can be risen by the imaginary character of the λ 's imposed by Eqs.(15); however, in explicit calculations these eigenvalues arise always as square roots of general functions which may have positive and negative values, depending on the parameters of the metric, giving place to real and imaginary λ 's. For instance, in our example of Eq.(7) we have that λ_1 is real, but λ_3 is imaginary. When an eigenvalue is real, the only solution to Eqs.(15) is that its associated n is equal to zero, but when the eigenvalue is imaginary the n can be any integer, leading to an infinite set of discrete solutions.

Now, let us consider the case in which the quantum conditions are not trivially satisfied, *i. e.*, Eqs.(15) hold for $n_1, n_2 \neq 0$. Then by using Eqs.(15) in Eq.(10) we obtain for A four different possibilities

$$A_1 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, A_2 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, A_3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \text{or} \quad A_4 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad (16)$$

where $\mathbb{1}$ is the 2×2 unit matrix. Inserting these expressions into Eq.(11), we obtain a phase-like object which is either the identity⁹

$$\Phi = T A_1 T^{-1} = \mathbb{1}_{4 \times 4}, \quad (17)$$

or something different from the identity for A_k ($k = 2, 3, 4$)

$$\Phi = T A_k T^{-1}, \quad (18)$$

an expression whose square, however, becomes the identity, *i. e.*,

$$\Phi^2 = T A_k T^{-1} T A_k T^{-1} = \mathbb{1}_{4 \times 4}, \quad (19)$$

⁸It is true as well that in general the λ 's depend on the coordinates which were not integrated in (4), but the independence of these coordinates could be required as an additional condition which would further specify the metrics that would lead to "realistic" quantum conditions.

⁹Since we have already imposed the condition $\Phi_1 = \Phi_2$, we can use indistinctly T_1 or T_2 for the following calculations.

where we have used the fact that $A_i^2 = \mathbb{1}_{4 \times 4}$, as can easily be verified from Eq.(16).

This is an interesting result, because when understood as an active diffeomorphism, the action of an observer traveling along a loop around an object is equivalent to rotating that object by 2π . Different authors have tried to model elementary particles by means of exact solutions of Einstein's equations [9–11,2,12]. If we assume that point of view here, we see that a particle described by a metric with the symmetry properties discussed above behaves under rotations only in two possible ways, as a boson in the case specified by Eq.(17), or as a fermion in the case specified by Eq.(18) and Eq.(19). Several investigations have been devoted to the search of this kind of behavior in models of elementary particles [11,3]. In this work we have found this behavior in gravitational configurations by using a very simple approach.

V. CONCLUDING REMARKS

We have shown that Dirac's argument about charge quantization in the presence of a magnetic monopole can be reproduced in the context of gravitation. We have defined a phase-like object in Eq.(4) that allows us to extract information about the behavior under rotations of any object enclosed within the surface of integration. In this direction further work has to be done with respect to the definition of the integral of the Riemann tensor over surfaces as an endomorphism valued two-form subject to a parallel transport. Although this definition is not given explicitly in the present work, some general conclusions have been stated, assuming that the symmetries of the system are respected by the final form of the integral.

We have used the normal form of the curvature tensor and its Petrov classification to show an invariant symmetry property given in Eq.(8) of the eigenvalues of a matrix which is constructed by a set of specific components of the curvature tensor, and enters the definition of the phase-like object (4). We then demand that the metrics under consideration be invariant with respect to a rotation by π of the azimuthal angle. This condition and the symmetry property (8) give enough information to calculate in terms of eigenvalues and eigenvectors the phase-like object Φ on each of the surfaces of integration, assuming that the the matrix integral I does not affect these symmetries. By imposing the quantization condition $\Phi_1 = \Phi_2$, we obtain an expression relating only the eigenvalues of the matrix integrals I_1 and I_2 . When the integration surfaces are homotopic, the quantization condition turns out to be trivially satisfied, whereas in the case of non homotopic surfaces we obtain conditions on the eigenvalues of the curvature tensor. These conditions indicate that the parameters entering the metric cannot be arbitrary, but they must be related by a discrete set of integer values. Additionally, we have shown that spacetimes satisfying the corresponding symmetry condition behave under rotations either as bosons or as fermions. First, it is interesting to see that gravitational configurations show this characteristic fermionic behavior even at the classical level. On the other hand, a quite remarkable result of our analysis is that in classical gravitational configurations no other possibilities are allowed, apart from bosonic or fermionic behavior.

Of course, to construct a realistic model of elementary particles many other aspects have to be considered, for instance, the requirement of the bosonic or fermionic behavior under the interchange of particles, the spin-statistics theorem [13–15], etc. Nevertheless, it is interesting to see that under this perspective one of these aspects is predicted correctly by the approach presented in the present work.

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